

Heterotic compactifications and nearly-Kähler manifolds

Andrei Micu[†]

*Department of Physics and Astronomy, University of Sussex
Brighton, BN1 9QH, United Kingdom.
email: A.Micu@sussex.ac.uk*

We propose that under certain conditions heterotic string compactifications on half-flat and nearly Kähler manifolds are equivalent. Based on this correspondence we argue that the moduli space of the nearly Kähler manifolds under discussion consists only of the Kähler deformations moduli space and there is no correspondent for the complex structure deformations.

Recently, a lot of attention was devoted to the study of compactifications on manifolds with reduced structure group. Such manifolds usually appear when one considers generalized compactifications in the presence of non-trivial fluxes. It is well-known that purely geometrical compactifications are supersymmetric if the internal manifold admits a covariantly constant spinor or in other words it has special holonomy [1]. When background fluxes are present these solutions are deformed away from the special holonomy limit and the internal manifold only has reduced structure group [2]–[16]. A second way to see these manifolds appearing is via dualities. In particular applying some duality transformation to a background which has non-trivial NS-NS fluxes generically leads to a deformed geometry [17]–[24]. The bottom line of both approaches is that in such compactifications a superpotential is generated and some of the geometric moduli get stabilized [4, 8, 9, 17, 19, 24, 25]. In a very recent work, [25], it was shown by an explicit calculation that for the case of heterotic strings compactified on manifolds with $SU(3)$ structure the superpotential has a very simple form¹

$$W = \sqrt{8} \int \Omega \wedge (H + idJ), \quad (1)$$

where Ω and J are the $SU(3)$ invariant forms on a manifold with $SU(3)$ structure and H is the field strength of the two-form field B . What is striking about this superpotential is that even if no particular assumption was done to obtain this expression, i.e. it is valid for *any* manifold with $SU(3)$ structure, the superpotential seems to depend explicitly only on the first torsion class \mathcal{W}_1 of the manifold under consideration (see Ref. [26] for a description of the five torsion classes of manifolds with $SU(3)$ structure).

In this note we plan to elaborate more on this issue, but do not keep the discussion general and instead concentrate on the half-flat manifolds which were considered in Ref. [25]. Note first that the Kähler potential does not depend on the torsion of the internal manifold and is the same as in the torsionless Calabi–Yau compactifications. Thus, if the above observation that the superpotential only depends on the torsion class \mathcal{W}_1 is true, it implies that the whole low energy physics depends just on this first torsion class and all the other torsion components are completely irrelevant from a four dimensional point of view. This is a curious statement, but also quite interesting as it allows us to find manifolds which are simpler than the ones considered in Ref. [25] and which produce the same result in four dimensions. The only requirement seems to be that the first torsion class, \mathcal{W}_1 , of these new manifolds is the same as the first torsion class of the manifolds considered in Ref. [25].

Among the manifolds with $SU(3)$ structure the simplest ones are the so called nearly Kähler manifolds (also known as manifolds with weak $SU(3)$ holonomy) for which only the torsion component in \mathcal{W}_1 is non-vanishing. On such manifolds the $SU(3)$ invariant forms J and Ω satisfy

$$dJ \sim \Omega, \quad (2)$$

and the value of the proportionality constant in this equation is given precisely by the torsion. Such manifolds have appeared recently in string compactification in Refs. [13, 15, 16] and it would be interesting to learn how to handle theories compactified on such manifolds. The first reason is that they are much simpler than the half-flat manifolds previously considered in the literature and moreover one has a better control on their geometry as the Ricci tensor and scalar can be computed explicitly. It is also worth noting that the half-flat manifolds discussed in

[†]On leave from IFIN-HH Bucharest.

¹See also Refs. [8, 9, 24].

Ref. [17] can not be reduced to nearly Kähler manifolds as in this case the torsion classes are related and one can not set them independently to zero. Thus, the question we will be focusing in this note is:

Does there exist a nearly Kähler manifold such that the compactification of the heterotic string at the lowest order in α' yields the same result as the one obtained in Ref. [25]?

To answer this question we would have to derive the low energy action of the heterotic string² compactified on a nearly Kähler manifold and compare it with what was obtained in Ref. [25].

Before we start there is a subtlety which we should clarify. The assertion that the superpotential (1) only depends on \mathcal{W}_1 is not quite true. To see this we integrate Eq. (1) by parts to obtain

$$W = \sqrt{8} \int d\Omega \wedge (B + iJ) . \quad (3)$$

By definition the term $\int d\Omega \wedge J$ is the component of the torsion in the first torsion class [26], but $\int d\Omega \wedge B$ can in general depend on the other torsion classes as well. Indeed, there is no reason for which B should be a singlet under $SU(3)$. There is however one exception to this, namely when there is only one Kähler modulus i.e. when the underlying a Calabi–Yau manifold has $h^{(1,1)} = 1$. Recall that the prescription for obtaining the moduli fields on a half-flat manifold is to expand the forms B and J in a basis of $(1, 1)$ forms. Thus we write

$$\begin{aligned} J &= v \omega , \\ B &= b \omega , \end{aligned} \quad (4)$$

and we see that B and J are actually proportional to each other. Consequently B is also a singlet under the structure group and so the superpotential (1) only depends on the torsion class \mathcal{W}_1 as we expected.

Restricting to this special case is not such a bad choice. First of all we know that there are examples of Calabi–Yau manifolds with $h^{(1,1)} = 1$, [27], and thus we can argue using the construction presented in Ref. [17] that the corresponding half-flat manifolds with the properties discussed above also exist. Moreover, many of the aspects of the heterotic strings compactified on Calabi–Yau manifolds are much easier handled for this special case of $h^{(1,1)} = 1$ and so we expect that the analysis in this note will not be completely irrelevant for this subject.

In this limit, the $N = 1$ supergravity obtained in Ref. [25] can be described in terms of the following Kähler potential

$$K = K_T + K_S + K^{(cs)} , \quad (5)$$

²In this note we do not consider at all the gauge fields and so the gauge group is irrelevant for the discussion.

where

$$K_T = -3 \ln(i(\bar{T} - T)) , \quad (6)$$

$$K_S = -\ln(i(\bar{S} - S)) , \quad (7)$$

$$K^{(cs)} = -\ln i \int \Omega \wedge \bar{\Omega} \equiv -\ln(\mathcal{V} \|\Omega\|^2) . \quad (8)$$

The superpotential has the form

$$W = \sqrt{8} e T , \quad (9)$$

and the complexified Kähler modulus is given as usual by $T = b + iv$. As argued before, this result only depends on the first torsion class \mathcal{W}_1 of the half-flat manifold³ and we would like to see what is the precise role of the other torsion classes on which this low energy action does not seem to depend. The way we propose to answer this question is by studying the compactification of the heterotic strings on a nearly Kähler manifold which would reproduce the above Kähler potential and superpotential. One of the major problem we face in this approach is the fact that the moduli space of manifolds with $SU(3)$ structure in general, and nearly Kähler manifolds in particular, is not known. On the other hand in the simpler case we are considering, namely that there is one single Kähler modulus,⁴ the expansions (4) are almost the only things one can write down. As a second argument for keeping the expansions (4) is that we would again like to think of these manifolds as small deformations of some underlying Calabi–Yau manifold on which one generically writes the moduli expansions (4).

It is important to note that once we make this choice, the Kähler potential and the superpotential are given by the general analysis presented in Ref. [25] and in particular they will have precisely the form (5) and (1). The only thing we have to fix is the torsion of this nearly Kähler manifold which should be chosen in such a way that the superpotential actually reproduces Eq. (9). As explained before, specifying the torsion for such manifolds amounts to make the relation (2) more precise. Since dJ is a real form, let us try⁵

$$dJ = \frac{2(ev)\Omega_-}{i \int \Omega \wedge \bar{\Omega}} , \quad (10)$$

³Note that in this case, the fact that there are no other $(1, 1)$ forms except ω which appears in (4), the torsion in the second class \mathcal{W}_2 also vanishes.

⁴Of course one can ask the question if such nearly Kähler manifolds do indeed exist. We are not aware of a precise answer to this question, but we know that one of the simplest examples of a nearly Kähler manifold is S^6 which definitely has this property. However, the sphere does not have only one spinor which is globally defined and so this example does not really fall into the class of manifolds we discuss in this paper.

⁵The choice of imaginary or real part of the form Ω is pure convention.

where Ω_- is the imaginary part of the form Ω and is defined as

$$\Omega_- = \frac{\Omega - \bar{\Omega}}{2i} . \quad (11)$$

Using this, and equation (4) we see that the basis form ω satisfies a similar relation

$$d\omega = \frac{2e\Omega_-}{i \int \Omega \wedge \bar{\Omega}} . \quad (12)$$

Inserting Eq. (4) in Eq. (1) and using Eq. (12) one can easily see that the superpotential is indeed given by (9).

To check that this set-up is correct we would have to compute the potential and see if it agrees with the one found in Ref. [25].⁶ For the computation which follows we use the conventions and the ten-dimensional action for the heterotic strings given in Ref. [25].

Again, like in Ref. [25], there will be two contributions to the potential: one coming from the kinetic term of the NS-NS field B in ten dimensions and the second from the curvature of the internal manifold. Let us compute them separately.

Using Eq. (12) the internal part of H is given by

$$H_{int} = 2(be) \frac{\Omega_-}{i \int \Omega \wedge \bar{\Omega}} . \quad (13)$$

By a standard calculation one immediately finds

$$\begin{aligned} V_H &= \frac{e^{2\phi}}{4\mathcal{V}} \int H_{int} \wedge *H_{int} \\ &= 4e^{2\phi+K_T+K^{cs}} (eb)^2 , \end{aligned} \quad (14)$$

where we used Eq. (8) and (11), and the fact that

$$8\mathcal{V} = i(T - \bar{T})^3 = e^{K_T} . \quad (15)$$

The computation of the scalar curvature for nearly Kähler manifolds is not difficult either. In fact one can compute the full Ricci tensor as these manifolds are known to be Einstein [28].

There are probably many ways to compute the curvatures of a nearly Kähler manifold, but here we will follow Ref. [29] where the Ricci tensor for manifolds with weak G_2 holonomy was computed by making use of the Killing spinor equation. To find this, note first that the (con)torsion of nearly Kähler manifolds is a totally antisymmetric tensor and using Eq. (10) one finds

$$\kappa_{mnp} = \frac{1}{3} e^{K^{(cs)}} (\Omega_+)_{mnp} . \quad (16)$$

⁶Here we implicitly assume that the kinetic terms of all the fields are the same as in the half-flat compactification presented in Ref. [25] and thus possible differences can come only from the potential.

Thus, the relation the $SU(3)$ invariant spinor⁷ η satisfies, is

$$\nabla_m \eta = \frac{e^{K^{(cs)}}}{12} (\Omega_+)_{mnp} \gamma^{np} \eta . \quad (17)$$

In order to compute the Ricci scalar using this relation it will be useful to expand the right hand side in the standard basis for spinors defined as

$$\eta , \quad \gamma_{\bar{\alpha}} \eta , \quad \gamma_{\alpha} \bar{\eta} , \quad \bar{\eta} , \quad (18)$$

where α denote complex indices which run $\alpha = 1, \dots, 3$. Noting that the right hand side of Eq. (17) has positive chirality, the most general expansion we can write is

$$\begin{aligned} \nabla_{\alpha} \eta &= \frac{e^{K^{(cs)}}}{12} (\Omega_+)_{\alpha\beta\gamma} \gamma^{\beta\gamma} \eta \\ &= A_{\alpha} \eta + A_{\alpha}^{\beta} \gamma_{\beta} \bar{\eta} . \end{aligned} \quad (19)$$

Multiplying successively by $\bar{\eta}^T$ and $\eta^T \gamma^{\delta}$ one obtains

$$A_{\alpha} = 0 \quad A_{\alpha}^{\beta} = i \frac{e^{K^{(cs)}} \sqrt{2}}{12} ||\Omega|| \delta_{\alpha}^{\beta} , \quad (20)$$

and thus, Eq. (17) becomes

$$\nabla_m \eta = i \frac{e^{K^{(cs)}} \sqrt{2}}{12} ||\Omega|| \gamma_m \eta . \quad (21)$$

From here on the calculation for the scalar curvature is straightforward. Taking the commutator of two covariant derivatives acting on the spinor η and using standard gamma-matrix relations the result is found to be [29]⁸

$$R = -\frac{5}{3} e^{2K^{(cs)}} (ev)^2 ||\Omega||^2 . \quad (22)$$

After integrating over the internal manifold one finds the contribution to the potential which comes from the curvature to be

$$\begin{aligned} V_R &= \frac{e^{2\phi}}{2\mathcal{V}} \int \sqrt{g} R \\ &= -\frac{20}{3} e^{2\phi+K^K+K^{(cs)}} (ev)^2 . \end{aligned} \quad (23)$$

Putting together the results from Eq. (14) and (23) we obtain

$$V = 4e^{2\phi+K^K+K^{(cs)}} \left[(eb)^2 - \frac{5}{3} (ev)^2 \right] . \quad (24)$$

Let us now compare this potential with the one derived in Ref. [25]. For the case $h^{(1,1)} = 1$ the inverse

⁷We use the convention that η is a Weyl spinor of positive chirality.

⁸Note that the different sign comes from the different convention we use for the Ricci scalar.

metric on the Kähler moduli space can be computed from the Kähler potential (6) and is given by $\frac{4}{3}v^2$. With this, the formula for the potential in Ref. [25] becomes

$$V_{hf} = 4e^{2\phi + K^K + K^{(cs)}} \left[4(eb)^2 + \frac{4}{3}(ev)^2 \right]. \quad (25)$$

Clearly the two formulae are different, but what had gone wrong because as we have argued at the beginning the low energy actions should have been the same as long as the first torsion classes were chosen to be the same. Thus we probably missed something in the above argument.

One of the things we have not discussed at all is the moduli spaces of half-flat and nearly Kähler manifolds. On the half-flat manifolds used in Ref. [25] one assumes that the moduli space of metrics is identical to the moduli space of some underlying Calabi–Yau manifold. Can this still be assumed for nearly Kähler manifolds as we implicitly did? The answer to this question is quite difficult as now we do not have any guiding principle in order to make such assumptions as it was the case for half-flat manifolds where mirror symmetry was enforcing the structure of the moduli spaces [17]. There is however one thing we can speculate on in the case of nearly Kähler manifolds. Equation (10) tells us that Ω_- is an exact form and thus its corresponding cohomology class is trivial.

For the moduli space of complex structures we were assuming that it can be again described in terms of the form Ω . However, if this is bound to be in the trivial class by the relation (10) it means that its variation with respect to the complex structure moduli can only be exact or in other words that nearly Kähler manifolds do not allow deformations which change Ω and thus they do not have what we called by abuse of language a complex structure moduli space.

This is quite intriguing so let us see if it can be independently verified by the low energy analysis we have presented so far. For this we would further have to specialise to a ‘rigid’ half-flat manifold which does not have any complex structure deformations. In Ref. [25] it was argued that the contribution to the potential from the complex structure moduli is $3|W|^2$. It is easy to see that subtracting this contribution from the potential (25) one indeed recovers the potential (24).

To summarize we have learned that the low energy action for the $(\alpha')^0$ sector of the heterotic string on a nearly Kähler manifold has the form

$$S_{nK} = \int \left\{ -\frac{1}{2}R * 1 - d\phi \wedge *d\phi - \frac{1}{4}da \wedge *da - \frac{3}{4v^2}dT \wedge *d\bar{T} - V * 1 \right\}, \quad (26)$$

with the potential V given in (24). Moreover the above comparison to the heterotic theory compacti-

fied on a half-flat manifold shows that this action is the bosonic part of an $N = 1$ supergravity coupled to two chiral multiplets S and T with Kähler potential

$$K = -\ln(i(\bar{S} - S)) - 3\ln(i(\bar{T} - T)), \quad (27)$$

and superpotential

$$W = \sqrt{8}eT. \quad (28)$$

Let us conclude this note by analyzing the main points derived so far. One of the most surprising results is that the (almost)complex structure of nearly Kähler manifolds is rigid and that there are no moduli associated to it. This is an interesting result in its own as it means that on such manifolds one only has to find a mechanism to fix the Kähler moduli. Moreover, proving this claim in general using a rigorous mathematical approach would be an important step forward in understanding the role of manifolds with $SU(3)$ structure in string compactifications. However, one has to keep in mind that in obtaining this result we assumed a couple of things which may restrict the validity of the argument. The action obtained in Eq. (26) corresponds to the compactification of the heterotic string on a half-flat manifold which has $h^{(2,1)} = 0$. The first problem comes from the fact that the half-flat manifolds were supposed to be the mirror duals of some Calabi–Yau manifold with NS-NS fluxes turned on and for this reason we need that $h^{(2,1)} > 0$. Ignoring this we still have to bare in mind that we were working in the limit $h^{(1,1)} = 1$ and it is not certain that a Calabi–Yau manifold with $h^{(1,1)} = 1$ and $h^{(2,1)} = 0$ does indeed exist. Furthermore, we were relying on the fact that the moduli space of nearly Kähler manifolds has a similar description to the moduli space of Calabi–Yau manifolds and splits into Kähler and complex structure deformations which in turn are governed by the $SU(3)$ singlet forms J and Ω respectively. Thus one of the minimal checks for the validity of the picture presented in this note would be to verify this claim in an independent mathematical way. On the other hand, despite these potential problems we believe that this is an important step forward in understanding more complicated (and realistic) compactifications on manifolds with $SU(3)$ structure and it would be interesting to extend the approach in this note to less constrained examples.

There are also a couple of interesting features which we should stress here. First it is definitely easier to deal with nearly Kähler manifolds and as we have seen one can compute in a transparent way their Ricci scalar. It is also worth noting that the expression obtained in (22) is negative and thus one can think of configurations where ordinary fluxes are turned on which contribute a positive energy density and can

balance the negative term coming from the curvature to give a Minkowski vacuum in four dimensions. Note that this possibility was not known before as the generic solution for compactifications on half-flat manifolds are domain walls [30] and in order to obtain a Minkowski vacuum additional constructions would be needed. However, it is important to stress here that even if such a ground state is found this should necessarily be non-supersymmetric as the conditions for supersymmetry derived in Ref. [31] require the internal manifold to be complex (see also Ref. [3] for a more recent discussion in terms of torsion classes). Last, but not least, it is worth noting that such manifolds can be an interesting arena for the study of an effective action in the presence of both electric and magnetic fluxes as this problem has not yet been solved. In Ref. [22] it was proved that the magnetic fluxes should come from non-vanishing derivatives of the form Ω_- . In the language of this paper this would correspond to a nearly Kähler manifold for which dJ in Eq. (10) has also piece proportional to Ω_+ . However we do not discuss this here anymore and we leave it for a further work.

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